5th Sensor Signal Processing for Defence Conference
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Low Complexity Parameter Estimation For Off-the-Grid Targets

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Objectives

- Estimate the radar parameters of a moving target, namely the reflection coefficient, angular location, and Doppler shift using the collocated MIMO-radar setting.
- Reduce the computational complexity of the algorithm by exploiting 2D-FFT to jointly estimate the angular location and Doppler shift.
- Enhance the resolution of the 2D-FFT estimates by applying a steepest decent algorithm to obtain off-the-grid estimates.

Outline:

1. System Model and Parameter Estimation
2. Iterative Method
3. Simulation Results

System Model:

- A MIMO radar system with uniform linear arrays at the transmitter and the receiver intercepts the following reflected signal:

$$\begin{aligned}\mathbf{y}(n) = & \overbrace{\beta_t e^{j2\pi f_{dt} n}\mathbf{a}_R(\theta_t)\mathbf{a}_T^T(\theta_t)\mathbf{x}(n)}^{\text{reflection from target}} \\ & + \underbrace{\sum_{i=1}^{L}\beta_i \mathbf{a}_R(\theta_i)\mathbf{a}_T^T(\theta_i)\mathbf{x}(n)}_{\text{reflection from interferers}} + \underbrace{\mathbf{v}(n)}_{\text{noise}},\ n = 1, 2, \ldots, N,\end{aligned} \tag{1}$$

where

- $\beta_t, \theta_t, f_{dt}$: the target's reflection coefficient, angular location, and Doppler shift.
- $\mathbf{a}_T(\theta_p)$, $\mathbf{a}_R(\theta_p)$: transmit and receive steering vectors at a location θ_p
- $\mathbf{x}(n)$: the vector of linearly independent symbols transmitted at time index n
- $\mathbf{v}(n)$: vector of complex white Gaussian noise samples

System Model:

- The Signal to Interference plus Noise Ratio (SINR) is maximized using the Capon beamformer $\mathbf{w}$ defined as

$$\mathbf{w}(\theta) = \frac{\mathbf{R}_{in}^{-1}\mathbf{a}_R(\theta)}{\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{a}_R(\theta)}, \tag{2}$$

where

$$\mathbf{R}_{in} = \mathrm{E}\left\{\left(\sum_{i=1}^{L}\beta_i\mathbf{a}_R(\theta_i)\mathbf{a}_T^T(\theta_i)\mathbf{x}(n)\right)\left(\sum_{i=1}^{L}\beta_i\mathbf{a}_R(\theta_i)\mathbf{a}_T^T(\theta_i)\mathbf{x}(n)\right)^H\right\} + \sigma_n^2\mathbf{I}_{n_R},$$

is the covariance matrix of the interference plus noise term. Using prior information of the interferers' parameters, the covariance matrix $\mathbf{R}_{in}$ can be computed. Otherwise, the methods proposed in [1,2] can be used to reconstruct it.

[1] Yujie Gu; Leshem, A., "Robust Adaptive Beamforming Based on Interference Covariance Matrix Reconstruction and Steering Vector Estimation," *IEEE Transactions on Signal Processing*, vol.60, no.7, pp. 3881-,3885, July 2012.

[2] Lei Huang; Jing Zhang; Xu Xu; Zhongfu Ye, "Robust Adaptive Beamforming With a Novel Interference-Plus-Noise Covariance Matrix Reconstruction Method," *IEEE Transactions on Signal Processing*, vol.63, no.7, pp. 1643-1650, April1, 2015

Parameter Estimation:

- To estimate the value of f_{dt}, θ_t, and β_t, the cost-function to be minimized can be written as

$$\{f_{dt}, \theta_t, \beta_t\} = \underset{f_d, \theta, \beta}{\text{argmin}} \ \mathrm{E}\left\{\left|\mathbf{w}^H(\theta)\mathbf{y}(n) - \beta e^{j2\pi f_d n}\mathbf{a}_T^T(\theta)\mathbf{x}(n)\right|^2\right\}. \quad (3)$$

- By differentiating the above cost-function with respect to β^* and equating it to 0, the minimizing value $\hat{\beta}$ can be found as

$$\hat{\beta}(f_d, \theta) = \frac{1}{n_T}\mathrm{E}\left\{e^{-j2\pi f_d n}\mathbf{w}^H(\theta)\mathbf{y}(n)\mathbf{x}^H(n)\mathbf{a}_T^*(\theta)\right\}. \quad (4)$$

- Thus, the cost function to be minimized in order to estimate f_{dt} and θ_t becomes

$$J_1(f_d, \theta) = \mathbf{w}^H(\theta)\mathbf{R}_y\mathbf{w}(\theta) - \frac{1}{n_T}\frac{\left|\mathrm{E}\left\{e^{-j2\pi f_d n}\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{y}(n)\mathbf{x}^H(n)\mathbf{a}_T^*(\theta)\right\}\right|^2}{\left|\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{a}_R(\theta)\right|^2}. \quad (5)$$

Parameter Estimation:

- In the absence of interferers, i.e., $\mathbf{R}_{in} = \sigma_n^2 \mathbf{I}_{n_R}$, minimizing J_1

$$J_1(f_d, \theta) = \mathbf{w}^H(\theta)\mathbf{R}_y\mathbf{w}(\theta) - \frac{1}{n_T}\frac{\left|\mathrm{E}\left\{e^{-j2\pi f_d n}\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{y}(n)\mathbf{x}^H(n)\mathbf{a}_T^*(\theta)\right\}\right|^2}{\left|\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{a}_R(\theta)\right|^2},$$

becomes equivalent to maximizing the following simplified cost function

$$J_2 = \left|\mathrm{E}\underbrace{\left\{e^{-j2\pi f_d n}\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{y}(n)\mathbf{x}^H(n)\mathbf{a}_T^*(\theta)\right\}}_{a(n)}\right|^2. \quad (6)$$

Parameter Estimation:

- Assuming $\mathbf{r}(n) = \mathbf{R}_{in}^{-1}\mathbf{y}(n)$, the term inside the expectation operator can be written as

$$\begin{aligned} a(n) &= e^{-j2\pi f_d n}\mathbf{a}_R^H(\theta)\mathbf{r}(n)\mathbf{x}^H(n)\mathbf{a}_T^*(\theta) \\ &= e^{-j2\pi f_d n}\sum_{p=1}^{n_T}\sum_{q=1}^{n_R} r_q(n)x_p^*(n)e^{-j2\pi\frac{d_R}{\lambda}(q-1)\sin(\theta)}e^{-j2\pi\frac{d_T}{\lambda}(p-1)\sin(\theta)} \\ &= e^{-j2\pi f_d n}\sum_{p=1}^{n_T}\sum_{q=1}^{n_R} r_q(n)x_p^*(n)e^{-j2\pi f_s(q-1+\gamma(p-1))}, \end{aligned} \quad (7)$$

where $f_s = \frac{d_R}{\lambda}\sin(\theta)$ and $\gamma = \frac{d_T}{d_R}$.

- By combining the same frequency terms, we can write

$$\mathrm{E}\{a(n)\} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{n_R-1+\gamma(n_T-1)} f(n,m)e^{-j2\pi f_d n}e^{-j2\pi f_s m}, \quad (8)$$

where $f(n,m) = \sum_{i=1}^{n_T} x_i^*(n)r_{m+1-\gamma(i-1)}(n)$.

- Exemple: If half wavelength ULA are used at the transmitter and receiver, i.e. $\gamma = 1$, we have

$$f(n,m) = \sum_{i=1}^{n_T} x_i^*(n) r_{m-i+2}(n). \tag{9}$$

$$\begin{array}{|c|c|c|c|c|}\hline x_1^*(n) & x_2^*(n) & x_3^*(n) & \cdots & x_{n_T}^*(n) \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline r_{n_R}(n) & \cdots & r_3(n) & r_2(n) & r_1(n) \\ \hline \end{array}$$

$$= f(n, m = 0)$$

- Therefore, the cost function J_2 can be computed using the 2D-FFT operator and the spatial and Doppler frequencies can be jointly estimated as follows

$$\hat{f}_{dt}, \hat{f}_{st} = \underset{f_d, f_s}{\operatorname{argmax}} \left| \sum_{n=0}^{N-1} \sum_{m=0}^{\substack{\gamma(n_T-1) \\ +n_R-1}} f(n,m) e^{-j2\pi f_d n} e^{-j2\pi f_s m} \right|^2 . \tag{10}$$

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$r_{n_R}(n)$	$\cdots$	$r_3(n)$	$r_2(n)$	$r_1(n)$			

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$$\begin{array}{|c|c|c|c|c|}\hline r_{n_R}(n) & \cdots & r_3(n) & r_2(n) & r_1(n) \\ \hline \end{array}$$

$$= f(n, m = 2)$$

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$$\hat{f}_{dt}, \hat{f}_{st} = \underset{f_d, f_s}{\operatorname{argmax}} \left| \sum_{n=0}^{N-1} \sum_{m=0}^{\substack{\gamma(n_T-1) \\ +n_R-1}} f(n,m) e^{-j2\pi f_d n} e^{-j2\pi f_s m} \right|^2. \tag{10}$$

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$x_1^*(n)$	$x_2^*(n)$	$x_3^*(n)$	$\cdots$	$x_{n_T}^*(n)$				
				$r_{n_R}(n)$	$\cdots$	$r_3(n)$	$r_2(n)$	$r_1(n)$

$$= f(n, m = n_T + n_R - 2)$$

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- Therefore, the cost function J_2 can be computed using the 2D-FFT operator and the spatial and Doppler frequencies can be jointly estimated as follows

$$\hat{f}_{dt}, \hat{f}_{st} = \underset{f_d, f_s}{\operatorname{argmax}} \left| \sum_{n=0}^{N-1} \sum_{m=0}^{\substack{\gamma(n_T-1)\\+n_R-1}} f(n, m) e^{-j2\pi f_d n} e^{-j2\pi f_s m} \right|^2. \tag{10}$$

Parameter Estimation:

- If interferers are present, as J_2 exactly estimates $\hat{f}_{dt}$, a search method is applied to find the estimate $\hat{\theta}_t$ that minimizes the cost function J_1,

$$J_1(\hat{f}_{dt}, \hat{\theta}_t) = \underset{\theta}{\operatorname{argmin}}\ \mathbf{w}^H(\theta)\mathbf{R}_y\mathbf{w}(\theta) - \frac{1}{n_T}\frac{J_2(\hat{f}_{dt}, \frac{d_R}{\lambda}\sin(\theta))}{\left|\mathbf{a}_R^H(\theta)\mathbf{R}_{in}^{-1}\mathbf{a}_R(\theta)\right|^2}.$$

- To reduce the computational cost, instead of evaluating the cost function J_1 over all grid points, we can restrict the search method over the region centered around the maximum of J_2.

• The low resolution estimates $\hat{f}_{dt}$ and $\hat{f}_{st}$ will be used to initialize the steepest decent algorithm and optimize the appropriate objective function J_1 or J_2 depending on whether the interferes are present or not.
• Thus, the first order derivatives with respect to θ and f_d of the following two expressions

$$J_2(\theta, f_d) = \frac{\left| \sum_{n=0}^{N-1} e^{-j2\pi f_d n} \mathbf{a}_R^H(\theta)\mathbf{r}(n)\mathbf{x}^H(n)\mathbf{a}_T^*(\theta) \right|^2}{N^2}, \tag{11}$$

and

$$A(\theta) = \mathbf{a}_R^H(\theta)\ \mathbf{G}\ \mathbf{a}_R(\theta), \tag{12}$$

are required. Here, $\mathbf{G}$ is a generic Hermitian positive semidefinite matrix of size n_R.

- Using matrix transformation, we can reformulate J_2 as

$$J_2(\theta, f_d) = \frac{1}{N^2}\left|\sum_{n=0}^{N-1} e^{-j2\pi f_d n}\mathbf{x}^H(n)\mathbf{M}_r(n)\mathbf{a}_S^*(\theta)\right|^2, \tag{13}$$

where the i^{th} row of the $n_T \times (n_R + \gamma(n_T - 1))$ matrix $\mathbf{M}_r(n)$ is defined as

$$(\mathbf{M}_r(n))_i = \left[\ \underbrace{0\cdots 0}_{(i-1)\gamma} \quad \mathbf{r}^T(n) \quad 0\cdots 0\ \right],\ i = 1, 2, \ldots, n_T,$$

$$\text{and } \mathbf{a}_S(\theta) = \left[1 \quad e^{j2\pi\frac{d_R}{\lambda}\sin(\theta)} \quad \ldots \quad e^{j2\pi\frac{d_R}{\lambda}\sin(\theta)(n_R-1+\gamma(n_T-1))}\right]^T.$$

- Considering the symmetry of $\mathbf{G}$, the expression $A(\theta)$ is reformulated as

$$A(\theta) = \sum_{l=1}^{n_R}\mathbf{G}_{l,l} + 2\Re\left(\sum_{l=1}^{n_R-1}\sum_{k>l}^{n_R}\mathbf{G}_{l,k}\mathrm{e}^{j2\pi f_s(k-l)}\right) = 2\Re\left(\mathbf{g}^T\mathbf{a}_R(\theta)\right), \tag{14}$$

where $\mathbf{g} = \left[\frac{1}{2}\sum_{l=1}^{n_R}\mathbf{G}_{l,l} \quad \sum_{l=1}^{n_R-1}\mathbf{G}_{l,l+1} \quad \cdots \quad \mathbf{G}_{1,n_R}\right]^T.$

- Hence, the first order derivatives of $J_2(\theta, f_d)$ with respect to f_d and θ are respectively

$$\frac{\partial J_2}{\partial f_d} = -\frac{4\pi}{N^2}\Im\left(\sum_{n=0}^{N-1} e^{-j2\pi f_d n}\mathbf{x}^H(n)\mathbf{M}_r(n)\mathbf{a}_S^*(\theta)\sum_{n=0}^{N-1} n e^{j2\pi f_d n}\mathbf{a}_S^T(\theta)\mathbf{M}_r^H(n)\mathbf{x}(n)\right), \tag{15}$$

and

$$\frac{\partial J_2}{\partial \theta} = -\frac{4\pi d_R \cos(\theta)}{\lambda N^2}\times$$
$$\Im\left(\sum_{n=0}^{N-1} e^{-j2\pi f_d n}\mathbf{x}^H(n)\mathbf{M}_r(n)\mathbf{a}_S^*(\theta)\sum_{n=0}^{N-1} e^{j2\pi f_d n}\mathbf{a}_S^T(\theta)\mathbf{D}_0^{\gamma(n_T-1)+n_R-1}\mathbf{M}_r^H(n)\mathbf{x}(n)\right). \tag{16}$$

- The first order derivative of $A(\theta)$ can be expressed as below

$$\frac{\partial A}{\partial \theta} = -4\pi\frac{d_R}{\lambda}\cos(\theta)\Im\left(\mathbf{g}^T\mathbf{D}_0^{n_R-1}\mathbf{a}_R(\theta)\right). \tag{17}$$

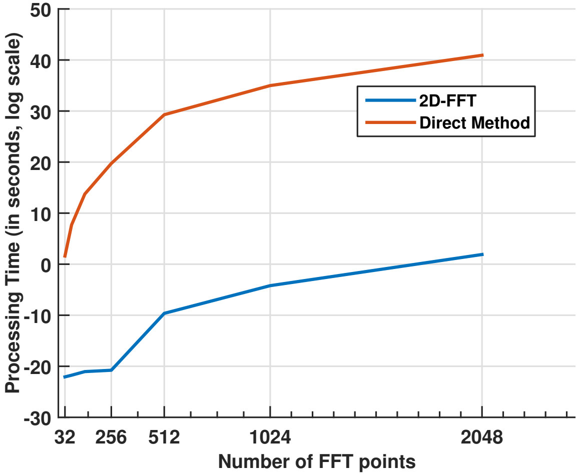

Fig. 1: Comparison of the processing time needed to compute the cost function J_2 for N=32 samples using 2D-FFT (blue) and direct method (red).

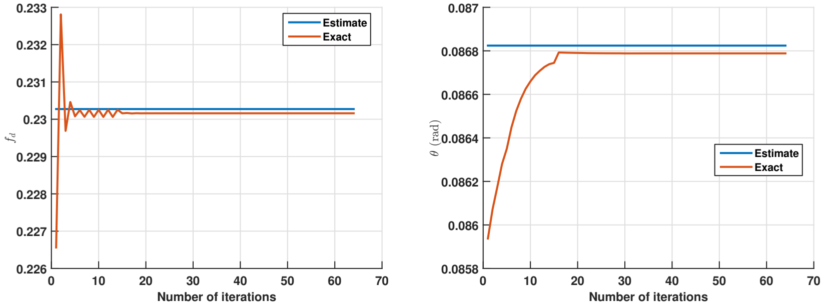

Fig. 2: Convergence behavior of the steepest descent algorithm for the estimation of the Doppler shift (left) and the spatial location (right) at SNR= 0 dB.

No Iterferers:

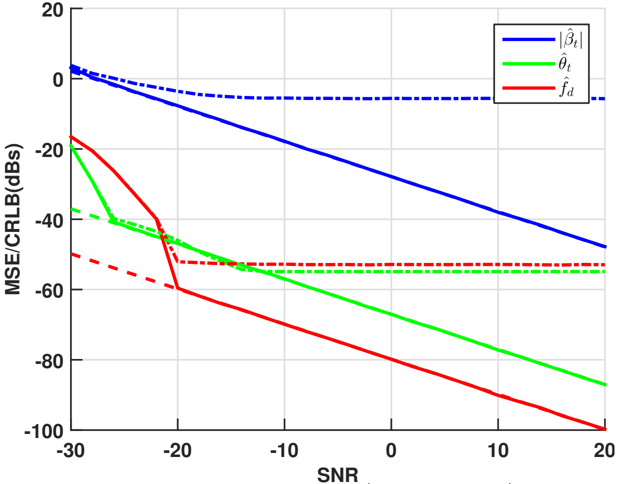

Fig. 3: Comparison of the 128 point 2D-FFT (dash-dot lines) and the iterative algorithm (solid lines) with the CRLB (dashed lines) of β_t, f_d, and θ_t. Here, $\beta_t = -1 + 2j$ and $\theta_t = 10°$.

With Iterferers:

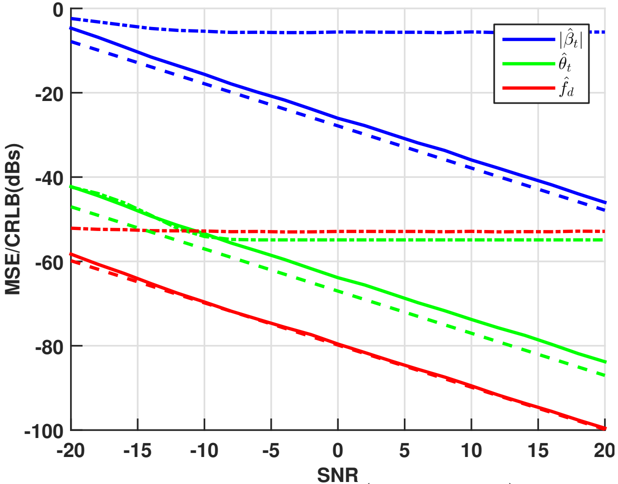

Fig. 4: Comparison of the 128 point 2D-FFT (dash-dot lines) and the iterative algorithm (solid lines) with the CRLB (dashed lines) of β_t, f_d, and θ_t. Here, $\beta_t = -1 + 2j$, $\theta_t = 10°$, and INR=20 dB.

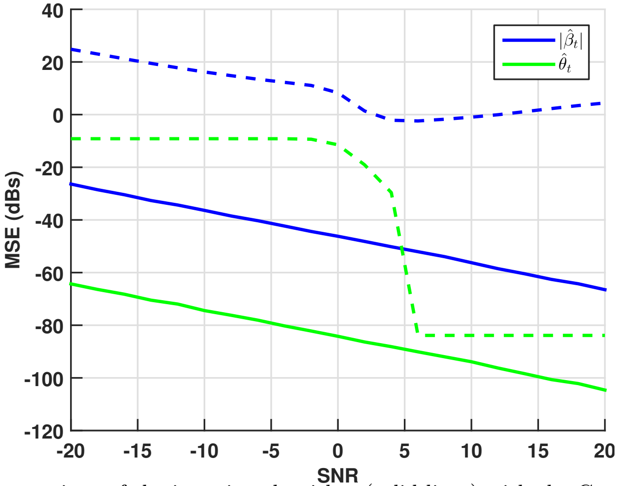

Fig. 5: Comparison of the iterative algorithm (solid lines) with the Capon (dashed lines) method derived in [**1**]. Here, $\beta_t = -1 + 2j$, $\theta_t = 10°$, and INR=20 dB.

[1] Luzhou Xu, Jian Li, and Petre Stoica, "Target detection and parameter estimation for MIMO radar systems," *IEEE Transactions on Aerospace and Electronic Systems*, vol.44, no. 3, pp. 927-939, July 2008.

Conclusion:

- An iterative algorithm estimates the reflection coefficient, the Doppler shift, and the spatial location of an off-the-grid target
- The low resolution 2D-FFT is used to find an initial point for the steepest descent algorithm
- The simulation results showed that the MSEE of the derived estimators matches the CRLB

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